Chapter 8 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 8: Differential Equations.

8.1: Slope Fields and Euler's Method
8.2: Separation of Variables
8.3: Exponential Models
8.4: Logistic Models
8.5: First-Order Linear Differential Equations

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

Differential Equations

Number of Questions—14

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points	Points Available
Short Questions		55
Question 12		15
Question 13		15
Question 14		15
TOTAL		**100**

Short Questions

1. For constants A and B, show that the function $y = \frac{7}{4} + Ae^{-4t} + Be^{-t}$ satisfies the second-order differential equation $y'' + 5y' + 4y = 7$. (5 pts.)

2. Use Euler's Method with step size 0.5 to approximate $y(1)$ if $\frac{dy}{dx} = 2x + y$ and $y(0) = 2$. (5 pts.)

3. Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y}\cos 4x$, $y\left(\frac{\pi}{4}\right) = 9$. (5 pts.)

4. Determine the general solution to the differential equation $8\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{2y}{t^2+1} = 0$. (5 pts.)

5. Find the orthogonal trajectories to the family of curves $y = kx^3$. *(5 pts.)*

6. A logistic decay model follows the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -0.04y(50-y)$. Identify the carrying capacity L and calculate the fastest rate of decrease of y. *(5 pts.)*

7. Determine the general solution to $y' + 4xy = e^{-2x^2}$.

8. The chemical reaction X $\longrightarrow$ Y is first-order. Initially, the concentration of X is 20 kilograms per liter. Three seconds after the reaction begins, the concentration of X decreases to 15 kilograms per liter. Calculate the concentration of X six seconds after the reaction begins. *(5 pts.)*

9. A function $g(x)$ grows at a rate proportional to the cube root of itself and inversely proportional to the cube of x. If $g(1) = 0$ and $g(2) = 1$, then determine the identity of g. *(5 pts.)*

10. A freshly baked pizza is removed from the oven at 250°C and allowed to cool at a room temperature of 20°C. Two minutes later, the pizza's temperature is measured to be 135°C. What is its temperature 4 minutes after being removed from the oven? *(5 pts.)*

11. A silo initially contains 50 kilograms of a chemical dissolved in 2000 liters of water. A solution whose concentration of 0.4 kilogram of chemical per liter of water enters the silo at a rate of 30 liters per minute. Simultaneously, the silo is drained and mixed well to maintain a constant volume of 2000 liters of water. Determine $f(t)$, the mass of chemical in the water t minutes after pumping begins. *(5 pts.)*

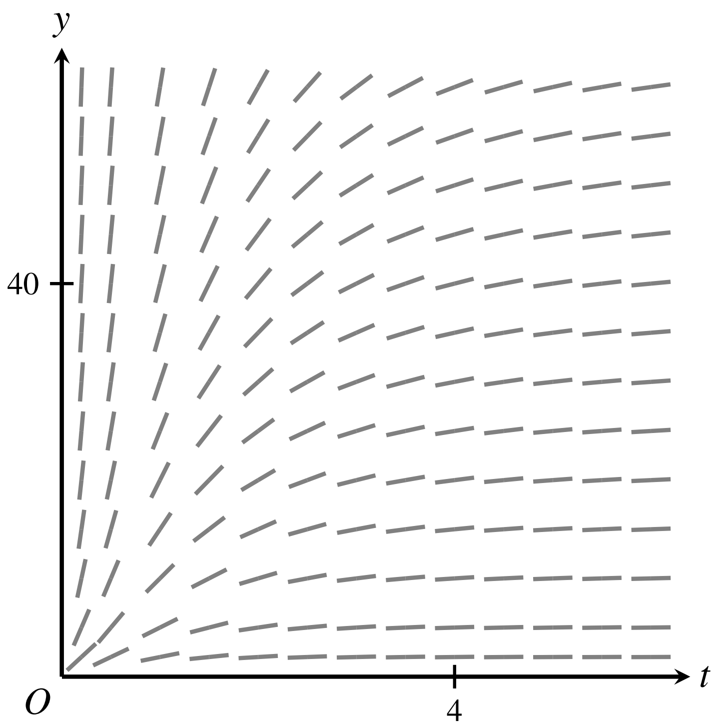
Long Questions

12. The number y of customers in a mall food court t hours after it opens is modeled by the differential equation

$$\frac{dy}{dt} = \frac{8y}{(t+1)^3}.$$

Four hours after opening, 40 customers are in the food court. Let $y = f(t)$ be the particular solution to the differential equation that satisfies the initial condition $f(4) = 40$.

(a) On the following slope field, sketch the graph of $y = f(t)$. (2 pts.)



(b) Does $f(t)$ exhibit logistic growth, exponential growth, or neither? Justify your answer. (2 pts.)

(c) Use Separation of Variables to find the identity of $f(t)$. *(5 pts.)*

(d) During the holiday season, more customers enter the food court. The mall models the number of shoppers in the food court using the new differential equation *(6 pts.)*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{8y}{(t+1)^3} + \frac{4}{(t+1)^5}.$$

There are 10 customers when the food court opens. Determine the particular solution $y = g(t)$.

13. A businessman has a principal amount of \$600 and is considering two investment plans. In a savings account at bank 1, the interest rate is 2%, compounded monthly. Bank 2 offers long-term bonds that generate 5% interest, compounded twice a year.

(a) Let $A_1(t)$ be the amount of money in a savings account at bank 1 after t years. Write an expression for $A_1(t)$. *(2 pts.)*

(b) Determine $A_2(t)$, the worth of a long-term bond at bank 2 after t years. *(2 pts.)*

(c) Bank 1 is considering altering its savings account to compound n times per year instead of every month. To what function does $A_1(t)$ approach as $n \to \infty$? *(3 pts.)*

(d) What differential equation does $A_1(t)$ satisfy as $n \to \infty$? (2 pts.)

(e) A third bank offers continuous compounding of money. A financial analyst proposes the differential equation $\frac{\mathrm{d}A}{\mathrm{d}t} = 1.03A + 600$ to model the rate of change of the account's balance $A(t)$ as a function of time in years. If the account has a principal balance of \$1000, then find the identity of $A(t)$. (6 pts.)

14. Ecologists discover a new colony of penguins and propose three different models to predict its future population, $N(t)$, as a function of time in years. There are 60 penguins at the time of discovery.

(a) The ecologists first believe that the population grows continuously at a rate of 4% per year. Write an equation for the population's proposed rate of growth, $\frac{\mathrm{d}N}{\mathrm{d}t}$, and find $N(t)$. *(4 pts.)*

(b) Write an expression for the doubling time of the model in part (a). *(1 pt.)*

(c) After observing some penguins emigrating from the colony, the ecologists change their proposed differential equation to $\frac{dN}{dt} = 0.06N - 2$. Using the initial condition $N(0) = 60$, solve this differential equation to determine the new model for $N(t)$. *(5 pts.)*

(d) Ten years after the initial discovery of 60 penguins, the colony grows to 200 penguins. Surprised by the stagnant growth, the ecologists now propose a logistic growth function with a carrying capacity of 300. Find this logistic function, $N(t)$. *(5 pts.)*

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions (5 points each)

1. Differentiating $y = \frac{7}{4} + Ae^{-4t} + Be^{-t}$ shows

$$y' = -4Ae^{-4t} - Be^{-t}\,, \quad *$$

$$y'' = 16Ae^{-4t} + Be^{-t}\,. \quad *$$

Substituting these expressions into $y'' + 5y' + 4y = 7$ shows

$$\underbrace{(16Ae^{-4t} + Be^{-t})}_{y''} + \underbrace{5(-4Ae^{-4t} - Be^{-t})}_{5y'} + \underbrace{4\left(\frac{7}{4} + Ae^{-4t} + Be^{-t}\right)}_{4y} \overset{\checkmark}{=} 7\,. \quad ***$$

2. The first iteration of Euler's Method is

$$y(0.5) \approx y(0) + h(2x+y)\Big|_{x=0\,,y=2} \quad *$$

$$= 2 + 0.5(2)$$

$$= 3\,. \quad *$$

The second iteration yields

$$y(1) \approx y(0.5) + h(2x+y)\Big|_{x=0.5\,,y=3} \quad *$$

$$= 3 + 0.5(4)$$

$$= \boxed{5} \quad **$$

3. Using Separation of Variables gives

$$\int \frac{\mathrm{d}y}{\sqrt{y}} = \int \cos 4x \,\mathrm{d}x \quad *$$

$$2\sqrt{y} = \tfrac{1}{4}\sin 4x + C \quad *$$

Substituting the initial condition $y\left(\frac{\pi}{4}\right) = 9$ shows

$$2\sqrt{9} = \tfrac{1}{4}\sin\pi + C \qquad *$$

$$\implies C = 6\,.$$

Hence, the solution is

$$2\sqrt{y} = \tfrac{1}{4}\sin 4x + 6\,. \qquad *$$

Solving for y gives

$$\boxed{y = \left(\tfrac{1}{8}\sin 4x + 3\right)^2} \qquad *$$

4. Using Separation of Variables, we attain

$$\int \frac{4\mathrm{d}y}{y} = \int \frac{1}{t^2+1}\,\mathrm{d}t \qquad *$$

$$4\ln|y| = \tan^{-1} t + C_1 \qquad *$$

$$|y| = e^{(\tan^{-1} t)/4} e^{C_1} \qquad *$$

Letting $C = \pm e^{C_1}$ gives

$$\boxed{y = Ce^{(\tan^{-1} t)/4}} \qquad **$$

5. Differentiating $y = kx^3$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2\,. \qquad *$$

But from $y = kx^3$, we get $k = \dfrac{y}{x^3}$, so the previous differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\left(\frac{y}{x^3}\right)x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y}{x}\,. \qquad *$$

The orthogonal trajectories have a slope equal to the negative reciprocal, as described by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{3y}. \quad *$$

Separation of Variables gives

$$\int y\,\mathrm{d}y = \int -\frac{x}{3}\,\mathrm{d}x \quad *$$

$$\tfrac{1}{2}y^2 = -\tfrac{1}{6}x^2 + C_1 \quad *$$

$$\boxed{x^2 + 3y^2 = C}$$

where $C = 6C_1$.

6. The graph of the logistic model is bounded above by the carrying capacity, so we seek the graph's larger horizontal asymptote. We determine the horizontal asymptotes by setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, as follows:

$$-0.04y(50-y) = 0 \quad *$$

$$\implies y = 0, 50. \quad *$$

We choose the larger solution to be $y = L$, which is

$$\boxed{L = 50} \quad *$$

The fastest rate of decrease occurs when $y = L/2 = 25$:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{y=25} = -0.04(25)(50-25) \quad *$$

$$= \boxed{-25} \quad *$$

7. This is a first-order linear differential equation in the form $y' + P(x)y = Q(x)$ with $P(x) = 4x$ and $Q(x) = e^{-2x^2}$. The integrating factor is

$$\mu(x) = e^{\int P(x)\,\mathrm{d}x} = e^{\int 4x\,\mathrm{d}x} = e^{2x^2}. \quad *$$

Multiplying both sides by $\mu(x) = e^{2x^2}$ gives

$$y'e^{2x^2} + 4xye^{2x^2} = 1\,. \qquad *$$

The left side is the Product Rule expansion of $(ye^{2x^2})'$, so we have

$$\left(ye^{2x^2}\right)' = 1\,. \qquad *$$

Integrating both sides, we get

$$ye^{2x^2} = x + C \qquad *$$

$$\implies \boxed{y = xe^{-2x^2} + Ce^{-2x^2}} \qquad *$$

8. Because the reaction is first-order, the concentration of X satisfies

$$[\mathrm{X}](t) = 20e^{-kt}\,, \qquad *$$

where 20 kg/L is the initial concentration of X. Substituting the initial condition $\mathrm{X}(3) = 15$ shows

$$20e^{-3k} = 15 \qquad *$$

$$e^{-3k} = \tfrac{3}{4}$$

$$\implies k = \tfrac{1}{3}\ln\tfrac{4}{3}\,. \qquad *$$

Therefore, the concentration of X with time is

$$[\mathrm{X}](t) = 20e^{-\left(\frac{1}{3}\ln\frac{4}{3}\right)t}$$

$$= 20\left(\tfrac{3}{4}\right)^{t/3} \qquad *$$

After 6 sec, the concentration becomes

$$[\mathrm{X}](6) = 20\left(\tfrac{3}{4}\right)^{6/3}$$

$$= \boxed{11.25\ \mathrm{kg/L}} \qquad *$$

9. If k is a proportionality constant, then $y = g(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k\sqrt[3]{y}}{x^3}\,. \qquad *$$

Using Separation of Variables, we have

$$\int \frac{\mathrm{d}y}{\sqrt[3]{y}} = \int \frac{k}{x^3}\,\mathrm{d}x \qquad *$$

$$\tfrac{3}{2}y^{2/3} = -\frac{k}{2x^2} + C\,.$$

From the initial condition $g(1) = 0$, we have

$$\tfrac{3}{2}(0)^{2/3} = -\frac{k}{2(1)^2} + C$$

$$\implies k = 2C\,. \qquad (1)$$

From the initial condition $g(2) = 1$,

$$\tfrac{3}{2}(1)^{2/3} = -\frac{k}{2(2)^2} + C$$

$$\implies 12 = -k + 8C\,. \qquad (2)$$

Substituting Equation (1) into Equation (2), we determine

$$C = 2 \qquad \text{and} \qquad k = 4\,. \qquad **$$

Thus, the solution is

$$\tfrac{3}{2}y^{2/3} = -\frac{2}{x^2} + 2$$

$$y = \left(-\frac{4}{3x^2} + \frac{4}{3}\right)^{3/2}$$

$$\boxed{g(x) = \left(-\frac{4}{3x^2} + \frac{4}{3}\right)^{3/2}} \qquad *$$

10. The initial temperature is $T_0 = 250^\circ$ C, and the ambient temperature is $T_s = 20^\circ$ C. By Newton's Law of Cooling, the pizza's temperature T satisfies

$$\begin{aligned} T(t) &= T_s + (T_0 - T_s)e^{-kt} \\ &= 20 + 230e^{-kt}, \quad * \end{aligned}$$

where t is time in minutes and k is some constant. Substituting the condition $T(2) = 135^\circ$ C shows

$$\begin{aligned} 20 + 230e^{-2k} &= 135 \quad * \\ 230e^{-2k} &= 115 \\ e^{-2k} &= \tfrac{1}{2} \\ \implies k &= \tfrac{1}{2}\ln 2\,. \quad * \end{aligned}$$

Thus, the temperature function becomes

$$\begin{aligned} T(t) &= 20 + 230e^{-\left(\frac{1}{2}\ln 2\right)t} \\ &= 20 + 230\left(\tfrac{1}{2}\right)^{t/2}. \quad * \end{aligned}$$

After 4 min, the pizza's temperature is

$$\begin{aligned} T(4) &= 20 + 230\left(\tfrac{1}{2}\right)^{4/2} \\ &= \boxed{77.5^\circ \text{ C}} \quad * \end{aligned}$$

11. The rate at which the chemical enters the water, in kilograms per minute, is

$$\text{rate in} = \frac{30 \text{ L}}{1 \text{ min}} \times \frac{0.4 \text{ kg}}{1 \text{ L}} = 12 \text{ kg/min}\,. \quad *$$

The silo's volume remains constant, so liquid simultaneously *exits* the tank at a rate of 30 L/min. Let y be the current mass of chemical in the tank. Accordingly, in terms of kilograms per minute,

$$\text{rate out} = \frac{30 \text{ L}}{1 \text{ min}} \times \frac{y \text{ kg}}{2000 \text{ L}} = \frac{3y}{200} \text{ kg/min}\,. \quad *$$

If t is time in minutes, then

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= 12 - \frac{3y}{200}\,. \\ &= \frac{2400-3y}{200}\,.\end{aligned}$$ *

Using Separation of Variables, we attain

$$\begin{aligned}\int \frac{dy}{2400-3y} &= \int \frac{dt}{200} \\ -\tfrac{1}{3}\ln|2400-3y| &= \frac{t}{200} + C\,.\end{aligned}$$ *

Substituting the initial condition $y(0) = 50$ shows

$$\begin{aligned}-\tfrac{1}{3}\ln 2250 &= 0 + C \\ \implies C &= -\tfrac{1}{3}\ln 2250\,.\end{aligned}$$

Accordingly, the solution is

$$\begin{aligned}-\tfrac{1}{3}\ln|2400-3y| &= \frac{t}{200} - \tfrac{1}{3}\ln 2250 \\ \ln|2400-3y| &= \ln 2250 - \frac{3t}{200} \\ |2400-3y| &= 2250e^{-3t/200}\end{aligned}$$

Only the positive solution in the absolute value expression adheres to the initial condition $y(0) = 50$, so use

$$\begin{aligned}2400 - 3y &= 2250e^{-3t/200} \\ \implies y &= 800 - 750e^{-3t/200}\,.\end{aligned}$$

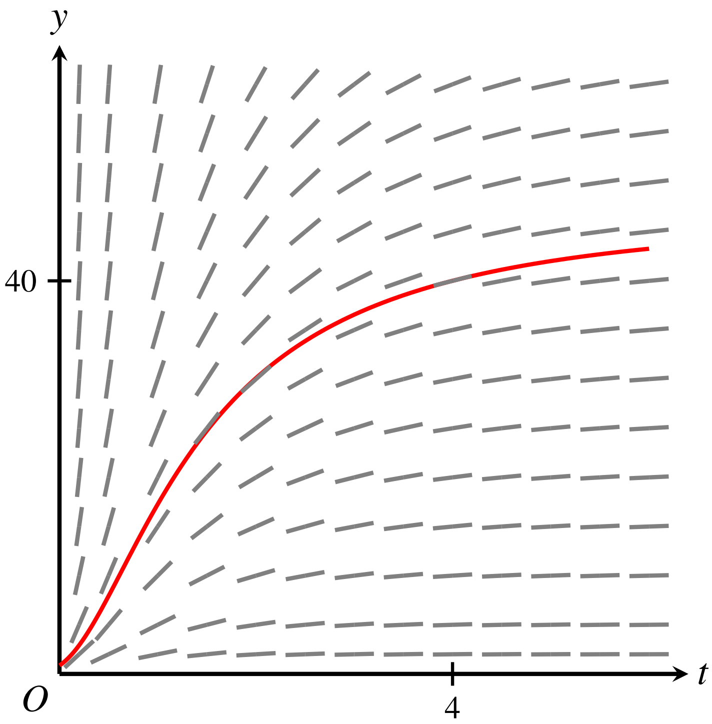
Hence,

$$\boxed{f(t) = 800 - 750e^{-3t/200}}$$ *

Long Questions (15 points each)

12. (a) The curve of the particular solution is as follows. The curve should

- pass through the point $(4,40)$ *
- follow the direction of the line segments *



(b) The differential equation's form is neither exponential growth, $y' = ky$, nor logistic growth, $y' = ky(1-y/L)$. Thus, the growth type is **

$$\boxed{\text{neither}}$$

(c) Using Separation of Variables, we have

$$\int \frac{\mathrm{d}y}{y} = \int \frac{8}{(t+1)^3}\,\mathrm{d}t$$ *

$$\ln|y| = -\frac{4}{(t+1)^2} + C.$$ *

Substituting the initial condition $y(4) = 40$ shows

$$\ln 40 = -\frac{4}{25} + C$$

$$\implies C = \frac{4}{25} + \ln 40. \quad *$$

Noting that $y > 0$, the solution is

$$\ln y = -\frac{4}{(t+1)^2} + \frac{4}{25} + \ln 40$$

$$\boxed{y = 40e^{4/25-4/(t+1)^2}} \quad **$$

(d) The differential equation can be rewritten in the form

$$\frac{dy}{dt} - \frac{8}{(t+1)^3}y = \frac{4}{(t+1)^5},$$

a first-order linear differential equation with $P(t) = -8/(t+1)^3$ and $Q(t) = 4/(t+1)^5$. Thus, the integrating factor is

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int -8/(t+1)^3\,dt} = e^{4/(t+1)^2}. \quad *$$

Multiplying both sides by $\mu(t) = e^{4/(t+1)^2}$ gives

$$\frac{dy}{dt}e^{4/(t+1)^2} - \frac{8}{(t+1)^3}ye^{4/(t+1)^2} = \frac{4}{(t+1)^5}e^{4/(t+1)^2}.$$

The left side is a Product Rule expansion:

$$\frac{d}{dt}\left(ye^{4/(t+1)^2}\right) = \frac{4}{(t+1)^5}e^{4/(t+1)^2}.$$

Integrating both sides, we attain

$$ye^{4/(t+1)^2} = \int \frac{4}{(t+1)^5}e^{4/(t+1)^2}\,dt. \quad *$$

Substituting $u = 4/(t+1)^2$, we have $\mathrm{d}u = -8/(t+1)^3\,\mathrm{d}t$. Therefore,

$$\int \frac{4}{(t+1)^5} e^{4/(t+1)^2}\,\mathrm{d}t = -\tfrac{1}{8}\int ue^u\,\mathrm{d}u$$ *

$$= -\tfrac{1}{8}ue^u + \tfrac{1}{8}e^u + C$$

$$= -\frac{e^{4/(t+1)^2}}{2(t+1)^2} + \frac{e^{4/(t+1)^2}}{8} + C.$$ *

Thus, the solution is

$$ye^{4/(t+1)^2} = -\frac{e^{4/(t+1)^2}}{2(t+1)^2} + \frac{e^{4/(t+1)^2}}{8} + C.$$ *

Substituting the initial condition $y(0) = 10$, we attain

$$10e^4 = -\frac{e^4}{2} + \frac{e^4}{8} + C$$

$$\implies C = \frac{83e^4}{8}.$$

Hence, we have

$$ye^{4/(t+1)^2} = -\frac{e^{4/(t+1)^2}}{2(t+1)^2} + \frac{e^{4/(t+1)^2}}{8} + \frac{83e^4}{8}$$

$$y = -\frac{1}{2(t+1)^2} + \frac{1}{8} + \frac{83}{8}e^{4-4/(t+1)^2}$$

The particular solution is therefore

$$\boxed{g(t) = -\frac{1}{2(t+1)^2} + \frac{1}{8} + \frac{83}{8}e^{4-4/(t+1)^2}}$$ *

13. (a) With $n = 12$ compounding periods per year, in addition to a principal balance of \$600 and interest rate of $r = 0.02$, we have

$$A_1(t) = \boxed{600\left(1 + \frac{0.02}{12}\right)^{12t}}$$ **

where t is time in years.

(b) With $n = 2$ compounding periods per year, in addition to a principal balance of \$600 and interest rate of $r = 0.05$, we have

$$A_2(t) = \boxed{600\left(1+\frac{0.05}{2}\right)^{2t}} \qquad **$$

where t is time in years.

(c) As $n \to \infty$, the balance approaches the formula for continuous compounding:

$$\lim_{n\to\infty} A_1(t) = \lim_{n\to\infty} 600\left(1+\frac{0.02}{n}\right)^{nt}$$
$$= \boxed{600e^{0.02t}} \qquad ***$$

(d) The function in part (c) exhibits exponential growth with rate constant $k = 0.02$, so it satisfies

$$\boxed{\frac{\mathrm{d}A_1}{\mathrm{d}t} = 0.02A_1} \qquad **$$

(e) Using Separation of Variables gives

$$\int \frac{\mathrm{d}A}{1.03A+600} = \int \mathrm{d}t \qquad *$$
$$\frac{1}{1.03}\ln|1.03A+600| = t + C. \qquad *$$

Substituting the initial condition $A(0) = 1000$ produces

$$\frac{1}{1.03}\ln 1630 = 0 + C$$
$$\implies C = \frac{\ln 1630}{1.03}. \qquad *$$

Hence, the solution (noting that $A > 0$) is

$$\frac{1}{1.03}\ln(1.03A+600) = t + \frac{\ln 1630}{1.03} \qquad *$$
$$\ln(1.03A+600) = 1.03t + \ln 1630$$
$$1.03A + 600 = 1630e^{1.03t}$$
$$\implies \boxed{A(t) = \frac{1630e^{1.03t} - 600}{1.03}} \qquad **$$

14. (a) The growth rate is $k = 0.04$, so the differential equation is

$$\boxed{\frac{dN}{dt} = 0.04N}$$ **

With the initial condition $N(0) = 60$, the identity of $N(t)$ is

$$N(t) = \boxed{60e^{0.04t}}$$ **

(b) The doubling time is

$$t_2 = \frac{\ln 2}{k} = \boxed{\frac{\ln 2}{0.04}}$$ *

(c) Using Separation of Variables gives

$$\int \frac{dN}{0.06N - 2} = \int dt$$ *

$$\frac{1}{0.06} \ln|0.06N - 2| = t + C.$$ *

Substituting the initial condition $N(0) = 60$, we get

$$\frac{1}{0.06} \ln|0.06(60) - 2| = 0 + C$$

$$\implies C = \frac{\ln 1.6}{0.06}.$$ *

Accordingly, assuming $N \geqslant 60$, the solution is

$$\frac{1}{0.06} \ln(0.06N - 2) = t + \frac{\ln 1.6}{0.06}$$ *

$$\ln(0.06N - 2) = 0.06t + \ln 1.6$$

$$0.06N - 2 = 1.6e^{0.06t}$$

$$3N - 100 = 80e^{0.06t}$$

$$\implies \boxed{N(t) = \frac{100}{3} + \frac{80}{3}e^{0.06t}}$$ *

(d) A logistic growth model with rate constant k and carrying capacity $L = 300$ takes the form

$$N(t) = \frac{300}{1 + Ce^{-kt}}\,, \qquad *$$

where C is a constant. Using the initial condition $N(0) = 60$ shows

$$\frac{300}{1 + C} = 60$$

$$\implies C = 4\,. \qquad *$$

Then substituting $N(10) = 200$ gives

$$\frac{300}{1 + 4e^{-10k}} = 200$$

$$1 + 4e^{-10k} = \tfrac{3}{2}$$

$$e^{-10k} = \tfrac{1}{8}$$

$$\implies k = \tfrac{1}{10}\ln 8\,. \qquad *$$

Hence, the solution is

$$N(t) = \frac{300}{1 + 4e^{-[(\ln 8)/10]t}}$$

$$= \boxed{\frac{300}{1 + 4\left(\frac{1}{8}\right)^{t/10}}} \qquad **$$